AD-A069 290 STANFORD UNIV CALIF DEPT OF OPERATIONS RESEARCH F/G 12/1

INSCRIBING AND CIRCUMSCRIBING CONVEX POLYHEDRA.(U)

MAR 79 B C EAVES, R M FREUND DAAG29-78-G-0026

UNCLASSIFIED ARO-15254.6-M NL

1 OF 1

AD

A069290

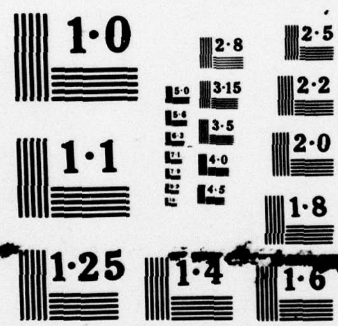
END

DATE

FILMED

7-79

DDC



NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

18 ARO

19 15254.6-M

6 **INSCRIBING AND CIRCUMSCRIBING CONVEX POLYHEDRA .**

by

10 B. Curtis Eaves [redacted] R. M. Freund

9 TECHNICAL [redacted]EP[redacted]T

11 Mar[redacted]79

12 14p.

Department of Operations Research
Stanford University
Stanford, California

15
This research ~~was sponsored in part by the~~ Army Research Office - Durham, Contract No DAAG 29-78-G-0026, and the National Science Foundation [redacted] MCS 77-05623

VNSF-

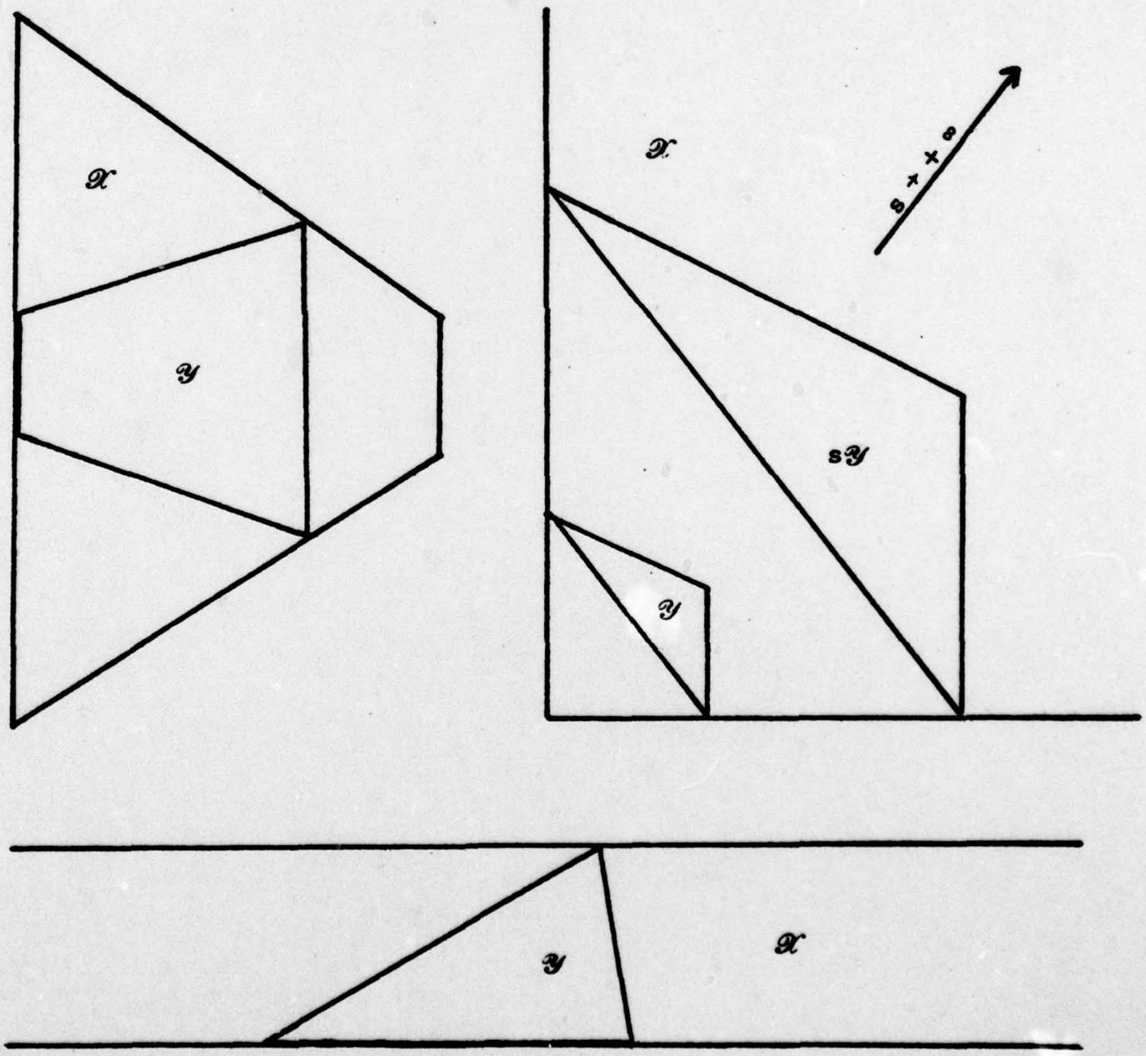
402 766

79 05 29 024

TABLE OF CONTENTS

1. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be polyhedra; that is, closed polyhedral convex sets, bounded or not, in R^n. Our interest is in computing the smallest nonnegative scale $s\mathscr{X}$ of $\mathscr{X}$ for which some translate $s\mathscr{X} + t$ contains $\mathscr{Y}$, or equivalently, of computing the largest nonnegative scale $s\mathscr{Y}$ of $\mathscr{Y}$ for which some translate $s\mathscr{Y} + t$ is contained in $\mathscr{X}$.



For certain descriptions of $\mathscr{X}$ and $\mathscr{Y}$ we observe that this problem; namely, the circumscription program

$$\text{P1}\quad \begin{cases} z_1 = \underset{s,t}{\text{infimum:}}\ \ s \\ \text{subject to:}\ \ \mathscr{Y} \subseteq s\mathscr{X} + t \qquad s \geq 0 \end{cases}$$

or equivalently, the inscription program

$$\text{P2}\quad \begin{cases} z_2 = \underset{s,t}{\text{supremum:}}\ \ s \\ \text{subject to:}\ \ s\mathscr{Y} + t \subseteq \mathscr{X} \qquad s \geq 0 \end{cases}$$

is a linear program.

That an n-sphere of maximum radius in a polyhedron can be found by linear programming has been in the folklore for over a decade and has been used in a variety of applications. Perhaps our observations will be of use as well.

2. Preliminaries

To avoid trivialities we assume throughout that $\mathscr{X}$ and $\mathscr{Y}$ contain at least one and two points, respectively.

$\mathscr{X}$ and $\mathscr{Y}$ can be described as the intersection of finitely many hyperplanes

$$\mathscr{X} = \{x \in R^n : Ax \leq a\} \qquad \mathscr{Y} = \{x \in R^n : Bx \leq b\}$$

in which case we say that $\mathscr{X}$ and $\mathscr{Y}$ have representation H . Alternatively, $\mathscr{X}$ and $\mathscr{Y}$ can be described as a weighting of points and rays

$$\mathscr{X} = \{x \in R^n : x = X\lambda + U\omega,\ e\lambda = 1,\ \lambda \geq 0,\ \omega \geq 0\}$$

$$\mathscr{Y} = \{x \in R^n : x = Y\mu + V\pi,\ e\mu = 1,\ \mu \geq 0,\ \pi \geq 0\}$$

where $e = (1, 1, \ldots 1)$ in which case we say $\mathscr{X}$ and $\mathscr{Y}$ have representation W .

Given a representation one can systematically convert it to the other; however, we shall suppose, and typically rightly so, that the computational burden of this conversion is prohibitive.

Using the representation W we see that any polyhedron $\mathscr{X}$ can be expressed as K + C where K is a compact polyhedron and C is the polyhedron of rays of $\mathscr{X}$. For any positive scale s we have $s\mathscr{X} = sK + sC = sK + C$, and, of course, for a zero scale s we have $s\mathscr{X} = \{0\}$. Thus the set $s\mathscr{X}$ is continuous in s for positive s, and is continuous in nonnegative s if and only if C contains only the origin.

Given $\mathscr{X}$ the cone of rays C is unique and we write $C = \text{cone}\,\mathscr{X}$. By $\text{tng}\,\mathscr{X}$ we mean the smallest subspace of R^n that contains a translate of $\mathscr{X}$.

The next lemmas describe the sense in which P1 and P2 are equivalent.

Lemma 1 (Principal Equivalence): If (s_1, t_1) is feasible or optimal for P1 and s_1 is positive, then $(s_2, t_2) = (1/s_1, -t_1/s_1)$ is feasible or optimal, respectively, for P2, and vice versa. ⊠

Lemma 2 (Feasibility): P1 is feasible if and only if $\text{cone}\,\mathscr{Y} \subseteq \text{cone}\,\mathscr{X}$ and $\text{tng}\,\mathscr{Y} \subseteq \text{tng}\,\mathscr{X}$. P2 is always feasible. ⊠

Lemma 3 (Attainment): The following are equivalent.

i) $0 < z_1 < +\infty$ ii) P1 has an optimum

iii) $0 < z_2 < +\infty$ iv) P2 has an optimum. ⊠

<u>Lemma 4</u> (Non-attainment): The following are equivalent:

i) $z_1 = 0$

ii) $z_2 = +\infty$

iii) $\mathscr{Y} + t \subseteq \text{cone}\,\mathscr{X}$ for some t $\boxtimes$

The possible discontinuity of $s\mathscr{X} + t$ at $s = 0$ accounts for the incomplete equivalence between P1 and P2 .

3. Results

In this section we formulate the circumscription and inscription problems P1 and P2 for three cases of representation; namely, HH, WW, and HW, where, for example, HW refers to $\mathscr{X}$ and $\mathscr{Y}$ having representations H and W, respectively.

In each case a more general problem is treated first. We generalize P1 and P2 to

$$\text{P3}\begin{cases}\inf_{\theta}\text{imum:} & c\theta \\ \text{subject to:} & \mathscr{Y} \subseteq \mathscr{X}(\theta) \qquad\qquad \theta \in \Theta\end{cases}$$

and

$$\text{P4}\begin{cases}\sup_{\theta}\text{remum:} & c\theta \\ \text{subject to:} & \mathscr{Y}(\theta) \subseteq \mathscr{X} \qquad\qquad \theta \in \Theta\end{cases}$$

where Θ is a polyhedron in R^k and $c\theta$ is a linear function of θ in Θ that measures some feature of the polyhedrons. We regain P1 and P2 from P3 and P4 by setting $\theta = (s,t)$, etc.

Case HH

Let $\mathscr{X}(\theta)$ be the set $\{x : A(\theta)x \leq a(\theta)\}$ where (A,a) is an affine function of θ in Θ and let $\mathscr{Y} = \{x : Bx \leq b\}$. The program P3 is seen to be, using an alternative theorem, the linear program

$$
\text{P5}\quad\begin{cases}\underset{\theta,\Lambda}{\text{minimize:}} & c\theta \\ \text{subject to:} & \Lambda B = A(\theta) \qquad \Lambda b \le a(\theta) \\ & \Lambda \ge 0 \qquad \theta \in \Theta\end{cases}
$$

Observing that $s\mathscr{X} + t = \{x : Ax \le sa + At\}$ for $s > 0$ and specializing P5 to solve P1 we obtain the linear program

$$
\text{P6}\quad\begin{cases} z_1 = \underset{s,t,\Lambda}{\text{minimize:}} & s \\ \text{subject to:} & \Lambda B = A \qquad \Lambda b \le as + At \\ & \Lambda \ge 0 \qquad s \ge 0\end{cases}
$$

<u>Case WW</u>

Let $\mathscr{Y}(\theta)$ be the set $\{Y(\theta)\mu + V(\theta)\pi : e\mu = 1,\ \mu \ge 0,\ \pi \ge 0\}$ where (Y,V) is an affine function of θ in Θ and let $\mathscr{X}$ be the set $\{X\lambda + U\omega : e\lambda = 1,\ \lambda \ge 0,\ \omega \ge 0\}$. The program P4 is the linear program

$$
\text{P7}\quad\begin{cases}\underset{\theta,\Lambda,\Omega,\Pi}{\text{maximize:}} & c\theta \\ \text{subject to:} & Y(\theta) = X\Lambda + U\Omega \qquad e\Lambda = e \\ & V(\theta) = U\Pi \qquad \theta \in \Theta \\ & \Lambda \ge 0 \qquad \Omega \ge 0 \qquad \Pi \ge 0\end{cases}
$$

Specializing P7 to solve P2 we obtain the linear program

$$
P8 \begin{cases} z_2 = \underset{s,t,\Lambda,\Omega,\Pi}{\text{maximize:}} \quad s \\ \text{subject to:} \quad t \circ e + sY = X\Lambda + U\Omega \qquad e\Lambda = e \\ \qquad sV = U\Pi \qquad s \geq 0 \\ \Lambda \geq 0 \qquad \Omega \geq 0 \qquad \Pi \geq 0 \end{cases}
$$

where $\circ$ denotes outer product. In solving P8 one first verifies that $V = U\Pi$ with $\Pi \geq 0$ has a solution, and then drops the constraints $sV = U\Pi$ and $\Pi \geq 0$.

Case HW_1

We treat the case HW twice as HW_1 and HW_2 where we approach the circumscription/inscription problems through P3 and P4, respectively. Let $\mathscr{X}(\theta) = \{x : A(\theta)\, x \leq a(\theta)\}$ where (A,a) is an affine function of θ in Θ and let $\mathscr{Y} = \{Y\mu + V\pi : e\mu = 1,\ \mu \geq 0,\ \pi \geq 0\}$. The program P3 is the linear program

$$
P9 \begin{cases} \underset{\theta}{\text{minimize:}} \quad c\theta \\ \text{subject to:} \quad A(\theta)Y \leq a(\theta) \circ e \\ \qquad A(\theta)V \leq 0 \qquad \theta \in \Theta \end{cases}
$$

Specializing P9 to solve P1 we obtain the linear program

$$\text{P10}\begin{cases} z_1 = \underset{s,t}{\text{minimize:}} & s \\ \text{subject to:} & AY \leq s a\circ e + A(t\circ e) \\ & AV \leq 0 \qquad s \geq 0 \end{cases}$$

In solving P10 one verifies $AV \leq 0$ and then drops the constraints $AV \leq 0$.

Case HW_2

Let $\mathscr{Y}(\theta)$ be the set $\{Y(\theta)\mu + V(\theta)\pi : e\mu = 1, \mu \geq 0, \pi \geq 0\}$ where (Y,V) is an affine function of θ in Θ and $\mathscr{X}$ is the set $\{x : Ax \leq a\}$. Then the program P4 is the linear program

$$\text{P11}\begin{cases} \underset{\theta}{\text{maximize:}} & c\theta \\ \text{subject to:} & AY(\theta) \leq a\circ e \\ & AV(\theta) \leq 0 \qquad \theta \in \Theta \end{cases}$$

Specializing P11 to solve P2 we obtain the linear program

$$\text{P12}\begin{cases} z_2 = \underset{s,t}{\text{maximize:}} & s \\ \text{subject to:} & A(sY + t\circ e) \leq a \\ & sAV \leq 0 \qquad s \geq 0 \end{cases}$$

In solving P12 one verifies $AV \leq 0$ and then drops the constraints $sAV \leq 0$. Observe that P11 remains a linear program if a also is an affine function of θ in Θ .

4. Related Problems

The forgoing raises the question as to whether the following problems can be solved as linear programs.

a) Case WH

b) Finding the largest $\mathcal{Y}$ in $\mathcal{X}$ or smallest $\mathcal{X}$ containing $\mathcal{Y}$ where rotations as well as scales and translations are permitted.

c) Finding the largest n-sphere $\mathcal{Y}$ in $\mathcal{X}$ where $\mathcal{X}$ has representation W .

d) Finding the smallest n-sphere $\mathcal{X}$ containing $\mathcal{Y}$ where $\mathcal{Y}$ has representation H or W .

We suspect, but have no comprehensive proofs, that none of these problems can be formulated as a linear program. Observe in a) that for fixed (X,U,B,b) Case WH can be formulated as a linear program by converting to one of the other cases; in b) the set of optimal $\mathcal{Y}$ may not be a convex set; and in d) the n-sphere may have an irrational radius given rational data.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) INSCRIBING AND CIRCUMSCRIBING CONVEX POLYHEDRA		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) B. Curtis Eaves and R. M. Freund		8. CONTRACT OR GRANT NUMBER(s) DAAG-29-78-G-0026
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research Stanford University Stanford, California 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematics Division U.S. Army Research Office Box CM, Duke Station, Durham, N.C. 27706		12. REPORT DATE March 1979
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE NA

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The findings of this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inscribing
Circumscribing
Polyhedra
Linear Programming

R superscript n

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\mathcal{X}$ and $\mathcal{Y}$ be closed polyhedral convex sets, bounded or not, in R^n. For certain representations of $\mathcal{X}$ and $\mathcal{Y}$ it is shown that the task of finding the smallest scale of $\mathcal{X}$ for which some translate contains $\mathcal{Y}$ can be resolved with linear programming.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)